Coordination Over Multi-Agent Networks With Unmeasurable States and Finite-Level Quantization

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Abstract—In this note, the coordination of linear discrete-time multi-agent systems over digital networks is investigated with unmeasurable states in agents' dynamics. The quantized-observer based communication protocols and Certainty Equivalence principle based control protocols are proposed to characterize the inter-agent communication and the cooperative control in an integrative framework. By investigating the structural and asymptotic properties of the equations of stabilization and estimation errors, which are nonlinearly coupled by the finite-level quantization scheme, some necessary conditions and sufficient conditions are given for the existence of such communication and control protocols to ensure the inter-agent state observation and cooperative stabilization. It is shown that these conditions come down to the simultaneous stabilizability and the detectability of the dynamics of agents and the structure of the communication network.

Index Terms—Cooperatability, finite-level quantization, multi-agent system, quantized observer.

I. INTRODUCTION

In recent years, the coordination of multi-agent systems has attracted lots of attention by the systems and control community due to its wide applications ([1]–[5]). For the coordination of multi-agent systems with digital networks, the inter-agent communication, which aims at obtaining neighbors' state information as precise as possible, is usually the foundation of designing the cooperative control laws. In real digital networks, communication channels only have finite capacities and the communication between different agents is a process which consists of encoding, information transmitting and decoding. For this case, the instantaneously precise communication is generally impossible and one may seek encoding-decoding schemes to achieve asymptotically precise communication.

The most basic coordination of multi-agent systems is distributed consensus or synchronization, which is also called cooperative stabilization [1]. Quantized consensus and consensus with quantized communication can be dated back to [6] and [7] with the static quantization. Carli *et al.* [8] proposed a dynamic encoding-decoding scheme for distributed averaging. They proved that with infinite-level logarithmic quantization, the closed-loop system can achieve exact average-consensus asymptotically. Li *et al.* [9] proposed a dynamic encoding-decoding scheme with vanishing scaling function and finite-level uniform quantizers. They proved that the exact average-consensus can be achieved exponentially fast based on merely one-bit information exchange per communication between neighbors. This algorithm was then further generalized to the cases with directed and time-varying topologies ([10], [11]), the case with time delays ([12]), the case with general linear agent dynamics with full measurable states ([19]) and the case with second-order integrator dynamics with partially measurable states ([16]). Recent works in this direction can be found in [13] for ternary information exchange, the continuous-time dynamics ([14]) and consensus over finite fields ([15]).

All the above literature ([6]–[16]) focused on designing specific communication and control protocols and analyzing the closed-loop performances for specific systems. However, a fundamental problem of the coordination of multi-agent systems over digital networks is for what kinds of dynamic networks, there exist proper communication and control protocols which can guarantee the objectives of the inter-agent communication and cooperative control jointly. The coordination of digital multi-agent networks consists of two fundamental factors, one is the inter-agent state observation by communication among agents, and the other one is the cooperative control by each agent to achieve given coordination objectives. The inter-agent state observation is the objective of the inter-agent communication and is the basis of designing the cooperative control laws. This is similar in spirit to that the state observation is the basis of the feedback control design for single-agent systems with unmeasurable states. It is of theoretical and practical significance to characterize the inter-agent state observation and the cooperative control of multi-agent systems in an integrative framework. In this framework, one needs to first give the conditions for the existence of communication and control protocols to ensure both the communication and control objectives. For the case with precise communication, the consentability of linear multi-agent systems were studied. The concept of consentability was first proposed by [17] and [18]. It was shown that the controllability of agent dynamics and the connectivity of the communication topology graph have a joint influence on the consentability. You and Xie [19] and Gu *et al.* [20] studied the consentability of single-input linear discrete-time systems and sufficient conditions were given with respect to (w. r. t.) relative state feedback control protocols in [19] and w. r. t. filtered relative state feedback control protocols in [20], respectively.

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In this note, motivated by [16]–[19], we consider the cooperatability of linear discrete-time multi-agent systems with unmeasurable states and finite communication data rate. We propose a class of communication protocols based on quantized-observer type encoders and decoders and a class of control protocols based on the relative state feedback control law and the Certainty Equivalence principle. The closed-loop dynamics of the cooperative stabilization and the state estimation errors are coupled by the nonlinearities generated by the finite-level quantization scheme. By investigating the structural and asymptotic properties of the overall closed-loop equations, we give some necessary conditions and sufficient conditions for achieving inter-agent state observation and cooperative stabilization jointly w. r. t. the proposed classes of communication and control protocols. It is shown that the cooperatability of multi-agent systems is related to the simultaneous stabilizability and the detectability of the dynamics of agents and the structure of the communication graph.

Different from [19] for the case with fully measurable states, we consider the case with unmeasurable states and the finite communication data rate. The quantized-observer type encoding/decoding scheme proposed for second-order integrators in [16] is generalized for the case with general linear dynamics. Compared with [19] and [20] which focused on sufficient conditions, we show that the simultaneous stabizability of $(A, \lambda_i(\mathcal{L})B)$, $i = 2, \cdots, N$ and the detectability of (A, C) are sufficient, and also necessary in some sense, for the cooperatability of the linear multi-agent systems over digital networks, where A, B and C are the system matrix, the input matrix and the output matrix, respectively, of each agent and $\lambda_i(\mathcal{L})$, $i = 2, \cdots, N$, are nonzero eigenvalues of the Laplacian matrix $\mathcal{L}$ of the communication graph. We also show that the stabilizability of (A, B)(detectability of (A, C)) is necessary for the cooperative stabilization (inter-agent state observation), regardless of whether the inter-agent state observation (cooperative stabilization) is required.

The following notation will be used. Denote the column vectors or matrices with all elements being 1 and 0 by $\mathbf{1}$ and $\mathbf{0}$, respectively. Denote the identity matrix with dimension n by I_n. Denote the sets of real numbers, positive real numbers and conjugate numbers by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{C}$, respectively, and $\mathbb{R}^n$ denotes the n-dimensional real space. For any given vector $X \in \mathbb{R}^n$ or matrix $X = [x_{ij}] \in \mathbb{R}^{n \times m}$, its transpose is denoted by X^T, and its conjugate transpose is denoted by X^*. Denote the Euclidean norm of X by $\|X\|$ and the infinite norm of X by $\|X\|_\infty$. Denote the kth element of vector X by $[X]_k$. Denote the spectral radius of square matrix X by $\rho(X)$. Define $\mathscr{B}_r^{n \times m} = \{X \in \mathbb{R}^{n \times m} \,|\, \|X\| < r\}$ and $\mathscr{B}_r^n = \{X \in \mathbb{R}^n \,|\, \|X\|_\infty < r\}$, $r \in \mathbb{R}^+ \bigcup \{+\infty\}$. The Kronecker product is denoted by $\otimes$.

II. PROBLEM FORMULATION

We consider a multi-agent network with N agent. The dynamics of the ith agent is given by

$$\begin{cases} x_i(t+1) = Ax_i(t) + Bu_i(t), \\ y_i(t) = Cx_i(t), \end{cases} \quad t = 0, 1, \cdots, \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$. Here, $x_i(t)$, $y_i(t)$ and $u_i(t)$ are the state, the output and the control input of agent i. The overall communication structure of the network is represented by a directed graph $\mathscr{G} = \{V, \mathscr{E}, \mathscr{A}\}$, where $V = \{1, \cdots, N\}$ is the node set and each node represents an agent; $\mathscr{E}$ denotes the edge set and there is an edge $(j, i) \in \mathscr{E}$ if and only if there is a communication channel from j to i, then, agent i is called the receiver and agent j is called the sender, or i's neighbor. The set of agent i's neighbors is denoted by $N_i = \{j \in V \,|\, (j, i) \in \mathscr{E}\}$. We denote $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ as the weighted adjacency matrix of $\mathscr{G}$, $a_{ij} > 0$ if and only if $j \in N_i$.

Here we assume $a_{ii} = 0$. Denote $deg_i = \sum_{j=1}^N a_{ij}$ as the in-degree of node i and $\mathscr{D} = diag(deg_1, \cdots, deg_N)$ is called the degree matrix of $\mathscr{G}$. The Laplacian matrix $\mathcal{L}$ of $\mathscr{G}$ is defined as $\mathcal{L} = \mathscr{D} - \mathscr{A}$, and its eigenvalues in an ascending order of real parts are denoted by $\lambda_1(\mathcal{L}) = 0$, $\lambda_i(\mathcal{L})$, $i = 2, \cdots, N$. The agent dynamics (1) together with the communication topology graph $\mathscr{G}$ is called a dynamic network[1] and is denoted by $(A, B, C, \mathscr{G})$.

For real digital networks, only finite bits of data can be transmitted at each time step, therefore, each agent needs to first quantize and encode its output into finite symbols before transmitting them. Each pair of adjacent agents uses an encoding-decoding scheme to exchange information: For each digital communication channel (j, i), there is an encoder/decoder pair, denoted by $H_{ji} = (\Theta_j, \Psi_{ji})$, associate with it. Here, Θ_j denotes the encoder maintained by agent j and Ψ_{ji} denotes the decoder maintained by agent i. For the dynamic network $(A, B, C, \mathscr{G})$, the set $\{H_{ji}, i = 1, \cdots, N, j \in N_i \,|\, H_{ji} = (\Theta_j, \Psi_{ji})\}$ of encoder-decoder pairs over the whole network is called a communication protocol, and the collection of such communication protocols is denoted by the communication protocol set $\mathscr{H}$.

In this note, we propose the following communication protocol set:

$$\mathscr{H}(\varrho, L_G) = \Big\{ H(\gamma, \alpha, \alpha_u, L, L_u, G), \gamma \in (0, \varrho), \alpha \in (0, 1],$$

$$\alpha_u \in (0, 1], L \in \mathbb{N}, L_u \in \mathbb{N}, G \in \mathscr{B}_{L_G}^{n \times p} \Big\}, \quad (2)$$

where $H(\gamma, \alpha, \alpha_u, L, L_u, G) = \{H_{ji}, i = 1, \cdots, N, j \in N_i \,|\, H_{ji} = (\Theta_j, \Psi_{ji}),\}$. Here the constants $L_G \in \mathbb{R}^+ \bigcup \{+\infty\}$, $\varrho \in (0, 1]$ are given parameters of the communication protocol set, while $\gamma, \alpha, \alpha_u, L, L_u$ and G are parameters of a specific communication protocol. For each digital channel (j, i), the encoder is given by

$$\Theta_j = \begin{cases} \hat{x}_j(0) = \hat{x}_{j0}, \ \hat{u}_j(0) = \hat{u}_{j0}, \\[4pt] s_j(t) = Q_{\alpha, L}\left(\dfrac{y_j(t-1) - C\hat{x}_j(t-1)}{\gamma^{t-1}} \right), \\[8pt] \hat{x}_j(t) = A\hat{x}_j(t-1) + \gamma^{t-1}Gs_j(t) + B\hat{u}_j(t-1), \\[6pt] \hat{u}_j(t) = \hat{u}_j(t-1) + \gamma^{t-1}s_{u,j}(t), \\[6pt] s_{u,j}(t) = Q_{\alpha_u, L_u}\left(\dfrac{u_j(t) - \hat{u}_j(t-1)}{\gamma^{t-1}} \right). \end{cases} \quad (3)$$

and the decoder is given by

$$\Psi_{ji} = \begin{cases} \hat{x}_{ji}(0) = \hat{x}_{j0}, \ \hat{u}_{ji}(0) = \hat{u}_{j0}, \\[4pt] \hat{x}_{ji}(t) = A\hat{x}_{ji}(t-1) + \gamma^{t-1}Gs_j(t) + B\hat{u}_{ji}(t-1), \\[6pt] \hat{u}_{ji}(t) = \hat{u}_{ji}(t-1) + \gamma^{t-1}s_{u,j}(t), \end{cases} \quad (4)$$

where $Q_{p,M}(\cdot)$ with $p \in (0, 1]$ and $M = 1, 2, \ldots$ is a finite-level uniform quantizer. For vector inputs, the definition is applied to each component.

$$Q_{p,M}(y) = \begin{cases} ip, & ip - \dfrac{p}{2} \le y < ip + \dfrac{p}{2}, \ i = 0, 1, \ldots, M-1 \\[8pt] Mp, & y \ge Mp - \dfrac{p}{2}, \\[8pt] -Q_{p,M}(-y), & y < -\dfrac{p}{2}. \end{cases}$$

At each time step t, agent j generates the symbolic data $s_j(t)$ and $s_{u,j}(t)$ by the encoder Θ_j and sends them to agent i through the channel (j, i). After $s_j(t)$, $s_{u,j}(t)$ are received, by the decoder Ψ_{ji}, agent i calculates $\hat{x}_{ji}(t)$ as an estimate of $x_j(t)$. Denote $E_{ji}(t) =$

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[1] The concept of dynamic network of agents without output equations was defined in [21].

$x_j(t) - \hat{x}_{ji}(t)$ as the state estimation error. From (3) and (4), $E_{ji}(t) = x_j(t) - \hat{x}_j(t)$, and is denoted by $E_j(t)$ for short. Here, we say that the dynamic network achieves inter-agent state observation if

$$\lim_{t \to \infty} (x_j(t) - \hat{x}_{ji}(t)) = \mathbf{0}, \ i = 1, \cdots, N, \ j \in N_i.$$

For the case with precise communication, Olfati-Saber and Murray [21] proposed a class of relative state feedback control protocols:

$$u_i(t) = K \sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)), \ i = 1, \cdots, N. \quad (5)$$

Based on (5) and the Certainty Equivalence principle, we propose the following control protocol set: $\mathscr{U}(L_K) = \{U(K), K \in \mathscr{B}_{L_K}^{m \times n}\}$, where

$$U(K) = \left\{ u_i(t), t = 0, 1, \cdots, i = 1, \ldots, N \Big| \right.$$
$$\left. u_i(t) = K \sum_{j \in N_i} a_{ij}(\hat{x}_{ji}(t) - \hat{x}_i(t)) \right\}. \quad (6)$$

The constant $L_K \in \mathbb{R}^+ \bigcup \{+\infty\}$ is the given parameter of the control protocol set and the gain matrix K is the parameter of a control protocol to be designed.

We say that the dynamic network $(A, B, C, \mathscr{G})$ is locally cooperatable if for any given positive constants C_1, C_2, C_3, there exist communication and control protocols $H \in \mathscr{H}$ and $U \in \mathscr{U}$, such that for any $x_i(0) \in \mathscr{B}_{C_1}^n$, $\hat{x}_{i0} \in \mathscr{B}_{C_2}^n$, and $\hat{u}_{i0} \in \mathscr{B}_{C_3}^m$, $i \in 1, \cdots, N$, the closed-loop system achieves inter-agent state observation and cooperative stabilization, that is,

$(a) \ \lim_{t \to \infty} E_j(t) = \mathbf{0}, \ j = 1, \cdots, N.$

$(b) \ \lim_{t \to \infty} (x_i(t) - x_j(t)) = \mathbf{0}, \ i, j = 1, \cdots, N.$

The dynamic network is called globally cooperatable if there exist communication and control protocols $H \in \mathscr{H}$ and $U \in \mathscr{U}$, such that for any given initial condition, the closed-loop system achieves inter-agent state observation and cooperative stabilization.

Remark 1: Different from [19] and [20], we consider the cooperatability of linear multi-agent systems with unmeasurable states and finite data rate. A quantized-observer based encoding-decoding scheme is proposed to estimate neighbors' states while decoding. From (4), the decoder has a similar structure as the Luenberger observer. For the case with precise communication, the quantizers degenerate to identical functions and the decoders degenerate to the Luenberger observers.

Remark 2: (i) Our quantized observer is based on the quantized innovation of $y_i(t)$ but not $y_i(t)$ itself. This type of observer is also called differential pulse code modulation (DPCM) in the communication community, which can save the bandwidth of the communication channel significantly ([9], [16]). (ii) In a single-agent system, the controller and observer are usually located on the same side, which means the exact value of the control input can be used to design the observer directly. However, for the inter-agent state observation of multi-agent systems, the observers are located faraway from the neighbors' controllers, which means the exact values of neighbors' control inputs are not available. Therefore, the estimations of neighbors' control inputs are added into our encoding-decoding schemes. (iii) For second-order integrator agents, the special dynamic structure makes it feasible to reconstruct neighbors' control inputs by differencing the delayed positions and velocities without explicitly estimate neighbors' control inputs ([16]) However, for general linear dynamics, the method in [16] can not be used. Here, we propose the Luenberger form decoders (4) with explicit estimations of neighbors' control inputs.

Remark 3: Here, as a preliminary research, the definition of cooperatability focus on the ability of multi-agent systems to achieve inter-agent state observation and cooperative stabilization. The cooperative stabilization (synchronization) is the most basic cooperation of multi-agent systems and forms the foundation of many other kinds of cooperative controls, such as formation and distributed tracking. The concept of cooperatability can be further expanded for more general coordination behaviors. One may wonder that to achieve synchronization, why we do not use the decentralized state feedback control law $u_i(t) = -Kx_i(t)$ for each agent, then all agents' states will go to zero without any inter-agent communication. We do not use the decentralized state feedback control law but (6) mainly for two points. (i) The decentralized state feedback control law leads to the trivial case, i. e. all agents' states will go to zero. Here, the closed-loop system can achieve more general behavior. One may see that all agents' states will approach the weighted average initial values multiplied by the exponent of the system matrix under the control protocol (6) (see also Remark 4). This gives more flexibility to achieve complex coordination behavior by adjusting control and system parameters. (ii) The control protocol (6) is more flexible than the decentralized state feedback control law. One may further extend it for the formation control based on relative state vectors ([22]):

$$u_i(t) = K \sum_{j \in N_i} a_{ij}(\hat{x}_{ji}(t) - \hat{x}_i(t) - b_{ij}), \ i = 1, \ldots, N, \quad (7)$$

or the distributed tracking problem:

$$u_i(t) = K_1 \sum_{j \in N_i} a_{ij}(\hat{x}_{ji}(t) - \hat{x}_i(t))$$
$$+ K_2 b_{i0}(\hat{x}_{0i}(t) - \hat{x}_i(t)), \ i = 1, \ldots, N. \quad (8)$$

III. MAIN RESULTS

In this section we give some necessary conditions and sufficient conditions which ensure $(A, B, C, \mathscr{G})$ to be cooperatable. The following assumptions will be used.

A1) There exists $K \in \mathbb{R}^{m \times n}$ such that the eigenvalues of $A - \lambda_i(\mathcal{L})BK$, $i = 2, \cdots, N$ are all inside the open unit disk of the complex plane.

A2) (A, C) is detectable.

Denote $\Delta_j(t-1) = \frac{y_j(t-1) - C\hat{x}_j(t-1)}{\gamma^{t-1}} - s_j(t)$ and $\Delta_{u,j}(t-1) = \frac{u_j(t-1) - \hat{u}_j(t-1)}{\gamma^{t-1}} - s_{u,j}(t)$ as the quantization errors of $Q_{\alpha,L}(\cdot)$ and $Q_{\alpha_u, L_u}(\cdot)$, respectively. Denote $\Delta(t) = (\Delta_1^T(t), \cdots, \Delta_N^T(t))^T$, $\Delta_u(t) = (\Delta_{u,1}^T(t), \cdots, \Delta_{u,N}^T(t))^T$. Denote $X(t) = (x_1^T(t), \cdots, x_N^T(t))^T$, $\hat{X}(t) = (\hat{x}_1^T(t), \cdots, \hat{x}_N^T(t))^T$, $U(t) = (u_1^T(t), \cdots, u_N^T(t))^T$, $\hat{U}(t) = (\hat{u}_1^T(t), \cdots, \hat{u}_N^T(t))^T$. Denote $E(t) = X(t) - \hat{X}(t)$, $H(t) = U(t) - \hat{U}(t)$, $\bar{X}(t) = (\frac{1}{\mathbf{1}^T \pi} \mathbf{1} \pi^T \otimes I_n)X(t)$, $\delta(t) = X(t) - \bar{X}(t)$, where π^T is the nonnegative left eigenvector w. r. t. the eigenvalue 0 of $\mathcal{L}$ and it can be verified that π^T has at least one nonzero element. Here, $\delta(t)$ is called the cooperative stabilization error. Denote the lower triangular Jordan canonical of $\mathcal{L}$ by $diag(0, J_2, \cdots, J_N)$ where J_i is the Jordan chain with respect to $\lambda_i(\mathcal{L})$. We know that there is $\Phi \in \mathbb{R}^{N \times N}$, consisting of the left eigenvectors and generalized left eigenvectors of $\mathcal{L}$, such that $\Phi \mathcal{L} \Phi^{-1} = diag(0, J_2, \cdots, J_N)$. Let $\Phi = (\pi, \phi_2, \cdots, \phi_N)^T$. Denote $\bar{J}(K) = I_{N-1} \otimes A - diag(J_2, \cdots, J_N) \otimes BK$, $J(G) = diag(A - GC, \cdots, A - GC)_{nN \times nN}$.

From (1), (3), (4) and (6), we have

$$X(t+1) = (I_N \otimes A)X(t) - (\mathcal{L} \otimes BK)\hat{X}(t). \quad (9)$$

and

$$E(t+1) = (I_N \otimes (A - GC))E(t) + (I_N \otimes B)H(t)$$
$$\gamma^t(I_N \otimes G)\Delta(t). \quad (10)$$

Note that $\pi^T \mathcal{L} = 0$, from (9) it is known that $\bar{X}(t+1) = (I_N \otimes A)\bar{X}(t)$. Thus, from (9), the definition of $E(t)$, $\delta(t)$ and noting that $\mathcal{L}\mathbf{1} = 0$, we have

$$\delta(t+1) = (I_N \otimes A - \mathcal{L} \otimes BK)\delta(t) + (\mathcal{L} \otimes BK)E(t). \quad (11)$$

Denote $F(t) = \frac{U(t+1)-\hat{U}(t)}{\gamma^t}$. By (3) and the definition of $H(t)$, we have

$$H(t+1) = U(t+1) - \hat{U}(t) - \gamma^t Q_{\alpha_u, L_u}\left(\frac{U(t+1)-\hat{U}(t)}{\gamma^t}\right)$$
$$= \gamma^t(F(t) - Q_{\alpha_u, L_u}(F(t))) = \gamma^t \Delta_u(t). \quad (12)$$

From (6), (10) and (11), we can see that

$$F(t) = (\mathcal{L} \otimes K - \mathcal{L} \otimes KA + \mathcal{L}^2 \otimes KBK)\frac{\delta(t)}{\gamma^t}$$
$$+ (\mathcal{L} \otimes K(A-GC) - \mathcal{L}^2 \otimes KBK - \mathcal{L} \otimes K)\frac{E(t)}{\gamma^t} \quad (13)$$
$$+ (I_{mN} + \mathcal{L} \otimes KB)\frac{H(t)}{\gamma^t} + (\mathcal{L} \otimes KG)\Delta(t).$$

Thus, we have the following equations:

$$\begin{cases} E(t+1) = (I_N \otimes (A - GC))E(t) + (I_N \otimes B)H(t) \\ \qquad\qquad + \gamma^t(I_N \otimes G)\Delta(t), \\ \delta(t+1) = (I_N \otimes A - \mathcal{L} \otimes BK)\delta(t) + (\mathcal{L} \otimes BK)E(t), \\ H(t+1) = \gamma^t(F(t) - Q_{\alpha_u, L_u}(F(t))), \\ F(t) = (\mathcal{L} \otimes K - \mathcal{L} \otimes KA + \mathcal{L}^2 \otimes KBK)\frac{\delta(t)}{\gamma^t} \\ \qquad + (\mathcal{L} \otimes K(A-GC) - \mathcal{L}^2 \otimes KBK - \mathcal{L} \otimes K)\frac{E(t)}{\gamma^t} \\ \qquad + (I_{mN} + \mathcal{L} \otimes KB)\frac{H(t)}{\gamma^t} + (\mathcal{L} \otimes KG)\Delta(t). \end{cases} \quad (14)$$

We have the following theorems. Due to space limit, the proofs of Theorems 1 and 2 are given in the form of sketch, whose details can be found in [25].

Theorem 1: For the dynamic network $(A, B, C, \mathscr{G})$, $L_G = +\infty$, $\varrho = 1$ and $L_K = +\infty$, suppose that Assumptions **A1)** and **A2)** hold. Then, for any given positive constants C_x, $C_{\hat{x}}$ and $C_{\hat{u}}$, there exist a communication protocol $H(\gamma, \alpha, \alpha_u, L, L_u, G) \in \mathscr{H}(\varrho, L_G)$ and a control protocol $U(K) \in \mathscr{U}(L_K)$ such that for any $X(0) \in \mathscr{B}_{C_x}^{nN}$, $\hat{X}(0) \in \mathscr{B}_{C_{\hat{x}}}^{nN}$ and $\hat{U}(0) \in \mathscr{B}_{C_{\hat{u}}}^{mN}$, the dynamic network $(A, B, C, \mathscr{G})$ achieves inter-agent state observation and cooperative stabilization under H and U, and there exist positive constants W and W_u independent of $\gamma, \alpha, \alpha_u, L, L_u, G$ and K, such that $\sup_{t\geq 0} \max_{1\leq j\leq N} \|\Delta_j(t)\|_\infty \leq W$ and $\sup_{t\geq 0} \max_{1\leq j\leq N} \|\Delta_{u,j}(t)\|_\infty \leq W_u$.

Sketch of Proof (i) Take $K \in \mathscr{B}_{+\infty}^{m \times n}$ which satisfies **A1**). (ii) Take $G \in \mathscr{B}_{+\infty}^{n \times p}$ such that $\rho(A - GC) < 1$. (iii) Select α, α_u and γ properly in $(0,1)$, take $L > \frac{1}{\alpha}L(K, G, \gamma, C_x, C_{\hat{x}}, C_{\hat{u}}, \alpha, \alpha_u) - \frac{1}{2}$ and $L_u > \frac{1}{\alpha_u}L_u(K, G, \gamma, C_x, C_{\hat{x}}, C_{\hat{u}}, \alpha, \alpha_u)-\frac{1}{2}$ where $L(K, G, \gamma, C_x, C_{\hat{x}}, C_{\hat{u}}, \alpha, \alpha_u)$ and $L_u(K, G, \gamma, C_x, C_{\hat{x}}, C_{\hat{u}}, \alpha, \alpha_u)$ are functions of the parameters. By mathematical induction, we can prove that the quantizers $Q_{\alpha,L}(\cdot)$ and $Q_{\alpha_u,L_u}(\cdot)$ are always unsaturate, which leads to uniformly bounded quantization errors. Then based

on the uniform boundedness of the quantization errors, one can prove that both the inter-agent state observation error and the cooperative stabilization error vanish exponentially fast. ∎

Remark 4: It can be verified that $\bar{X}(t+1) = (I_N \otimes A)\bar{X}(t)$, $t = 0, 1, 2, \cdots$. Since $\lim_{t\to\infty} \delta(t) = 0$, we have

$$\lim_{t\to\infty}\left[x_i(t) - A^t\left(\frac{\sum_{i=1}^N \pi_i x_i(0)}{\sum_{i=1}^N \pi_i}\right)\right] = 0, \quad i = 1, \cdots, N.$$

So all agents' states will finally approach the trajectory $A^t\left(\frac{\sum_{i=1}^N \pi_i x_i(0)}{\sum_{i=1}^N \pi_i}\right)$. If the control protocol (6) is replaced by

$$u_i(t) = K_1 x_i(t) + K_2 \sum_{j\in N_i} a_{ij}(\hat{x}_{ji}(t) - \hat{x}_i(t)),$$
$$t = 0, 1, \cdots, i = 1, \ldots, N. \quad (15)$$

which combines the decentralized state feedback and the distributed quantized relative output feedback, then the closed-loop states approach $(A + BK_1)^t\left(\frac{\sum_{i=1}^N \pi_i x_i(0)}{\sum_{i=1}^N \pi_i}\right)$. For this kind of control protocols, one may choose K_1 to achieve more complex coordination behavior.

Remark 5: Intuitively, Assumption **A1)** contains the requirement on the agent dynamics (A, B) and the communication topology graph $\mathscr{G}$. If $\rho(A) < 1$, cooperative stabilization can be achieved by taking $K = \mathbf{0}$ (leading to the trivial case), which makes $A - \lambda_i(\mathcal{L})BK = A$, $i = 2, \cdots, N$ all stable even $\mathscr{G}$ has no spanning tree ($\lambda_2(\mathcal{L}) = 0$). If $\rho(A) \geq 1$, then Assumption **A1)** requires that $\lambda_2(\mathcal{L}) \neq 0$, which implies that $\mathscr{G}$ contains a spanning tree [23].

For single input discrete-time systems, [19] gave a necessary and sufficient condition to ensure **A1)** if all of A's eigenvalues are on or outside the unit circle of the complex plane, which was a intuitional explanation of **A1)**. In fact, for single input agents, a sufficient condition to ensure **A1)** can be given:

A1$'$) (A, B) is stabilizable and

$$\prod_j |\lambda_j^u(A)| < \frac{1}{\inf_{\omega\in\mathbb{R}} \max_{j\in\{2,\cdots,N\}} |1 - \omega\lambda_j(\mathcal{L})|}.$$

Here, $\lambda_j^u(A)$, $1 \leq j \leq n$ denote the unstable eigenvalues of A. If $\rho(A) < 1$, then $\prod_j |\lambda_j^u(A)|$ is defined as 0. What's more, if the communication topology graph is undirected, it was shown in [19] that $\frac{1}{\inf_{\omega\in\mathbb{R}} \max_{j\in\{2,\cdots,N\}} |1-\omega\lambda_j(\mathcal{L})|} = \frac{1+\lambda_2/\lambda_N}{1-\lambda_2/\lambda_N}$ and thus the eigenvalue-ratio λ_2/λ_N plays an important part in the cooperatability of linear multi-agent systems.

Theorem 2: For single input agents, if Assumption **A1$'$)** holds, then Assumption **A1)** holds.

Sketch of Proof For the case of $\rho(A) \geq 1$, it can be proved by reduction to absurdity that $\lambda_j(\mathcal{L}) \neq 0$, $j = 2, \cdots, N$. Without loss of generality, we can assume that matrix A has a block diagonal form $diag(A_s, A_u)$ where $\rho(A_s) < 1$ and all the eigenvalues of A_u are outside the open unit disk of the complex plane. If not, we can use a invertible matrix to transform A into that block diagonal form. Respectively, we have $B = [B_1^T, B_2^T]^T$. Since $\lambda_i(\mathcal{L}) \neq 0, i = 2, \cdots, N$, from Theorem 3.2 of [19], we know that there exist a $\tilde{K}$ such that $\rho(A_u - \lambda_i(\mathcal{L})B_2\tilde{K}) < 1, i = 2, \cdots, N$. Take $K = [\mathbf{0}^T, \tilde{K}]$, then we can see that $\rho(A - \lambda_i(\mathcal{L})BK) < 1$, $i = 2, \cdots, N$. For the case of $\rho(A) < 1$, we can see that **A1$'$)** suffices for **A1)** by takeing $K = \mathbf{0}$. ∎

Theorem 1 shows that Assumptions **A1)** and **A2)** are sufficient conditions for the cooperatability of $(A, B, C, \mathscr{G})$. Furthermore, we find that they are also necessary conditions if $\varrho < 1$.

Theorem 3: For $(A, B, C, \mathscr{G})$ and $L_G > 0$, $L_K > 0$ and $\varrho \in (0,1)$, suppose that for any given positive constants C_x, $C_{\hat{x}}$ and $C_{\hat{u}}$, there exist a communication protocol $H(\gamma, \alpha, \alpha_u, L, L_u, G) \in \mathscr{H}(\varrho, L_G)$ and a control protocol $U(K) \in \mathscr{U}(L_K)$, such that for any $X(0) \in \mathscr{B}_{C_x}^{nN}$, $\hat{X}(0) \in \mathscr{B}_{C_{\hat{x}}}^{nN}$ and $\hat{U}(0) \in \mathscr{B}_{C_{\hat{u}}}^{mN}$, the closed-loop system achieves inter-agent state observation and cooperative stabilization under H and U, and the quantization errors satisfy $\sup_{t\geq 0} \max_{1 \leq j \leq N} \|\Delta_j(t)\|_\infty \leq W$ and $\sup_{t\geq 0} \max_{1 \leq j \leq N} \|\Delta_{u,j}(t)\|_\infty \leq W_u$, where W and W_u are positive constants independent of $\gamma, \alpha, \alpha_u, L, L_u, G$ and K. Then Assumptions **A1)** and **A2)** hold.

Proof: We use reduction to absurdity. Select a constant a satisfying

$$a > \frac{4W_u\|B\|\sqrt{mN}}{1-\varrho} + \frac{4L_G W\sqrt{nN}}{1-\varrho}. \tag{16}$$

Take $C_x > \sqrt{n(2N-1)}a\|\Phi^{-1}\|$, $C_{\hat{x}} > \sqrt{nN}C_x + a\sqrt{nN}$ and $C_{\hat{u}} > \sup_{K \in \mathscr{B}_{L_K}} \|\mathcal{L} \otimes K\|C_{\hat{x}}\sqrt{nN}$. Now we prove that if **A1)** or **A2)** would not hold, then for such C_x, $C_{\hat{x}}$ and $C_{\hat{u}}$ and any communication protocol in (2) and control protocol in (6), there exist $X(0) \in \mathscr{B}_{C_x}^{nN}, \hat{X}(0) \in \mathscr{B}_{C_{\hat{x}}}^{nN}$ and $\hat{U}(0) \in \mathscr{B}_{C_{\hat{u}}}^{mN}$ such that the dynamic network can not achieve inter-agent state observation and cooperative stabilization jointly, which leads to the contradiction.

Denote $\tilde{\delta}(t) = (\Phi \otimes I_n)\delta(t)$. Denote $\bar{\Phi} = (\phi_2, \cdots, \phi_N)^T$, and denote $\tilde{\delta}_2(t) = (\bar{\Phi} \otimes I_n)\delta(t)$. From (14), it follows that

$$\begin{pmatrix} E(t+1) \\ \tilde{\delta}_2(t+1) \end{pmatrix} = A(K,G)\begin{pmatrix} E(t) \\ \tilde{\delta}_2(t) \end{pmatrix} + \begin{pmatrix} I_{nN} \\ \mathbf{0} \end{pmatrix}(I_N \otimes B)$$
$$\cdot H(t) + \begin{pmatrix} I_{nN} \\ \mathbf{0} \end{pmatrix}(I_N \otimes G)\gamma^t \Delta(t), \tag{17}$$

where $A(K,G) = \begin{pmatrix} J(G) & \mathbf{0} \\ (\bar{\Phi} \otimes I_n)(\mathcal{L} \otimes BK) & \bar{J}(K) \end{pmatrix}$. Since **A1)** and **A2)** would not hold simultaneously, we have $\rho(A(K,G)) \geq 1$ under any communication protocol in (2) and control protocol in (6). Transform $A(K,G)$ to its Schur canonical, that is, select a unitary matrix P ($P^* = P^{-1}$) such that

$$P^*A(K,G)P$$
$$= \begin{pmatrix} \lambda_1(A(K,G)) & & & \mathbf{0} \\ \times & \ddots & & \\ \times & \times & \lambda_{(2N-1)n}(A(K,G)) \end{pmatrix}.$$

Here, $\lambda_1(A(K,G))$, $\cdots$, $\lambda_{(2N-1)n}(A(K,G))$ are eigenvalues of $A(K,G)$ with $|\lambda_1(A(K,G))| = \rho(A(K,G))$, and $\times$ represents the elements below the diagonal of the Schur canonical.

Denote $Z(t) = P^*[E^T(t), \tilde{\delta}_2^T(t)]^T$. From (17) we know that

$$[Z(t+1)]_1$$
$$= \lambda_1^{t+1}(A(K,G))[Z(0)]_1$$
$$+ \sum_{i=1}^{t} \lambda_1^{t-i}(A(K,G))\left[P^*[I_{nN}, \mathbf{0}^T]^T(I_N \otimes B)H(i)\right]_1$$
$$+ \sum_{i=0}^{t} \lambda_1^{t-i}(A(K,G))\gamma^i\left[P^*[I_{nN}, \mathbf{0}^T]^T(I_N \otimes G)\Delta(i)\right]_1$$
$$+ \lambda_1^t(A(K,G))\left[P^*[I_{nN}, \mathbf{0}^T]^T(I_N \otimes B)H(0)\right]_1. \tag{18}$$

Let $P = [P_1^T, P_2^T]^T$ with $P_1 \in \mathbb{R}^{nN \times n(2N-1)}$ and $P_2 \in \mathbb{R}^{n(N-1) \times n(2N-1)}$. Take $X(0) = (\Phi^{-1} \otimes I_n)[\mathbf{0}^T, a^T P_2^T]^T$ where $\mathbf{a} = a\mathbf{1} \in \mathbb{R}^{n(2N-1)}$ and $\mathbf{0} \in \mathbb{R}^n$, then $\|X(0)\|_\infty \leq \sqrt{n(2N-1)}a\|\Phi^{-1}\|\|P_2\|$. Note that $\|P_2\| \leq \|P\| = 1$, we have

$\|X(0)\|_\infty \leq \sqrt{n(2N-1)}a\|\Phi^{-1}\| < C_x$, implying $X(0) \in \mathscr{B}_{C_x}^{nN}$. Take $\hat{X}(0) = X(0) - P_1\mathbf{a}$ and $\hat{U}(0) = -(\mathcal{L} \otimes K)\hat{X}(0)$. Similarly, one can see that $\hat{X}(0) \in \mathscr{B}_{C_{\hat{x}}}^{nN}$ and $\hat{U}(0) \in \mathscr{B}_{C_{\hat{u}}}^{mN}$. By the definition of $\delta(t)$ and some direct calculation, we have $\tilde{\delta}(0) = [\mathbf{0}^T, \mathbf{a}^T P_2^T]^T$, and $\tilde{\delta}_2(0) = P_2\mathbf{a}$. By the definition of $E(t)$ and $H(t)$, we know that $E(0) = X(0) - \hat{X}(0) = X(0) - (X(0) - P_1\mathbf{a}) = P_1\mathbf{a}$, and $H(0) = U(0) - \hat{U}(0) = -(\mathcal{L} \otimes K)\hat{X}(0) + (\mathcal{L} \otimes K)\hat{X}(0) = \mathbf{0}$. Since $Z(0) = P^*[E(0)^T, \tilde{\delta}_2^T(0)]^T$, we have $Z(0) = \mathbf{a}$ and $[Z(0)]_1 = a$.

From (16), we know that

$$\left| \sum_{i=1}^{t} \lambda_1^{t-i}(A(K,G))\left[P^*[I_{nN}, \mathbf{0}^T]^T(I_N \otimes B)H(i)\right]_1 \right.$$
$$\left. + \sum_{i=0}^{t} \lambda_1^{t-i}(A(K,G))g(i)\left[P^*[I_{nN}, \mathbf{0}^T]^T(I_N \otimes G)\Delta(i)\right]_1 \right|$$
$$\leq \left(\frac{2W_u\|B\|\sqrt{mN}}{1-\varrho} + \frac{2L_G W\sqrt{nN}}{1-\varrho}\right)|\lambda_1(A(K,G))|^{t+1}$$
$$< \frac{a}{2}|\lambda_1(A(K,G))|^{t+1}. \tag{19}$$

From (18), (19) and noting that $H(0) = \mathbf{0}$, we have

$$\left|[Z(t+1)]_1\right| \geq \left| |\lambda_1(A(K,G))|^{t+1}a - \frac{a}{2}|\lambda_1(A(K,G))|^{t+1} \right|$$
$$= \frac{a}{2}|\lambda_1(A(K,G))|^{t+1}.$$

By the invertibility of P, we know that $[E^T(t), \delta^T(t)]^T$ does not vanish as $t \to \infty$. This is in contradiction with that the dynamic network achieves inter-agent state observation and cooperative stabilization. So, **A1)** and **A2)** hold. ∎

Remark 6: Actually, the communication protocol parameter γ can represent the convergence speed of the cooperative coordination (for both inter-agent state observation and cooperative stabilization). The smaller γ is, the faster the convergence will be. The constant ϱ is an upper bound of γ, so it is a uniform upper bound of the convergence speed. Theorem 3 shows that if $(A, B, C, \mathscr{G})$ is locally cooperatable with a uniform exponential convergence speed, then **A1)** and **A2)** hold.

Remark 7: Sundaram and Hadjicostis ([24]) showed that a linear system is structurally controllable and observable over a finite field if the graph of the system satisfies certain properties and the size of the field is large enough. They also applied this result into the control of multi-agent systems over finite fields. Compared with [24], this note has the following differences. (i) [24] focused on the controllability and observability of linear systems over finite fields, and the closure property of the finite field plays an important role in getting their results. In this note we study the quantized coordination of linear multi-agent systems over real number field, so the closure and invertible properties can not be used. (ii) The system matrix A of the linear system in [24] corresponds to the graph structure of the network and the dynamics of each agent is actually in some integrator form. What is more, the elements of the system matrices A, B and C are restricted in finite fields. In this note, the affect of the graph topology is decided by the Laplacian matrix, and each agent has the general linear dynamics (see (1)), where the system matrices A, B and C are arbitrary real matrices.

As preliminary research, this note is concerned with inter-agent state observation and cooperative stabilization of multi-agent systems over digital networks. It is an interesting topic for further investigation that whether our results can be combined with the methodology of [24] to study the controllability of multi-agent networks under quantized communication.

At present, we still do not know whether **A1)** and **A2)** are necessary conditions for $(A, B, C, \mathscr{G})$ to be locally cooperatable w. r. t. $\mathscr{H}(1, +\infty)$ and $\mathscr{U}(+\infty)$. However, we can show that if $(A, B, C, \mathscr{G})$ is globally cooperatable, then **A1)** and **A2)** are necessary w. r. t. $\mathscr{H}(1, +\infty)$ and $\mathscr{U}(+\infty)$.

Theorem 4: For $(A, B, C, \mathscr{G})$ and $L_G = +\infty$, $L_K = +\infty$ and $\varrho = 1$, if there exist a communication protocol $H(\gamma, \alpha, \alpha_u, L, L_u, G) \in \mathscr{H}(\varrho, L_G)$ and a control protocol $U(K) \in \mathscr{U}(L_K)$, such that for any $X(0) \in \mathbb{R}^{nN}$, $\hat{X}(0) \in \mathbb{R}^{nN}$ and $\hat{U}(0) \in \mathbb{R}^{mN}$, the closed-loop system achieves inter-agent state observation and cooperative stabilization under H and U, and $\sup_{t \geq 0} \max_{1 \leq j \leq N} \|\Delta_j(t)\|_\infty < \infty$ and $\sup_{t \geq 0} \max_{1 \leq j \leq N} \|\Delta_{u,j}(t)\|_\infty < \infty$, then Assumptions **A1)** and **A2)** hold.

Proof: Denote $\sup_{t \geq 0} \max_{1 \leq j \leq N} \|\Delta_j(t)\|_\infty$ by W and $\sup_{t \geq 0} \max_{1 \leq j \leq N} \|\Delta_{u,j}(t)\|_\infty$ by W_u. Noting that here, different from Theorem 3, W and W_u may depend on the parameters γ, α, α_u, L, L_u, G and K. Select a constant a greater than $\frac{4\|B\|\sqrt{mN}W_u}{|\lambda_1(A(K,G)) - \gamma|} + \frac{4\|G\|W\sqrt{nN}}{|\lambda_1(A(K,G)) - \gamma|}$. Take $X(0) = (\Phi^{-1} \otimes I_n)[\mathbf{0}^T, \mathbf{a}^T P_2^T]^T$ where $\mathbf{a} = a\mathbf{1} \in \mathbb{R}^{n(2N-1)}$ and $\mathbf{0} \in \mathbb{R}^n$. Take $\hat{X}(0) = X(0) - P_1\mathbf{a}$, $\hat{U}(0) = -(\mathcal{L} \otimes K)\hat{X}(0)$, thus $E(0) = X(0) - \hat{X}(0) = X(0) - (X(0) - P_1\mathbf{a}) = P_1\mathbf{a}$, and $H(0) = U(0) - \hat{U}(0) = -(\mathcal{L} \otimes K)\hat{X}(0) + (\mathcal{L} \otimes K)\hat{X}(0) = \mathbf{0}$. Thus $Z(0) = \mathbf{a}$ and $[Z(0)]_1 = a$. Then similar to the proof of Theorem 3, we have the conclusion. ∎

From the following theorems, we can see that the stabilizability of (A, B) is necessary for $(A, B, C, \mathscr{G})$ to achieve cooperative stabilization no matter whether the inter-agent state observation is required, and similarly, the detectability of (A, C) is necessary for $(A, B, C, \mathscr{G})$ to achieve inter-agent state observation regardless of the cooperative stabilization.

Theorem 5: For $(A, B, C, \mathscr{G})$, $L_G = +\infty$, $L_K = +\infty$ and $\varrho = 1$, suppose that for any given positive constants C_x, $C_{\hat{x}}$ and $C_{\hat{u}}$, there exist a communication protocol $H(\gamma, \alpha, \alpha_u, L, L_u, G) \in \mathscr{H}(\varrho, L_G)$ and a control protocol $U(K) \in \mathscr{U}(L_K)$, such that for any $X(0) \in \mathscr{B}_{C_x}^{nN}$, $\hat{X}(0) \in \mathscr{B}_{C_{\hat{x}}}^{nN}$ and $\hat{U}(0) \in \mathscr{B}_{C_{\hat{u}}}^{mN}$, the closed-loop system achieves cooperative stabilization under H and U, that is, $\lim_{t \to \infty}(x_j(t) - x_i(t)) = \mathbf{0}$, $\forall i, j = 1, 2, \ldots, N$. Then (A, B) is stabilizable.

Proof: We use the reduction to absurdity to prove this theorem. Suppose that (A, B) is unstabilizable, then there exists an invertible matrix T_1, such that $T_1^{-1}AT_1 = \begin{pmatrix} A_{s_1} & A_{12} \\ \mathbf{0} & A_{u4} \end{pmatrix}$ and $T_1^{-1}B = (B_3^T, \mathbf{0}^T)^T$, where $A_{u4} \in \mathbb{R}^{n_{u4} \times n_{u4}}$ is unstable. Here n_{u4} is a positive integer. Take $C_x > \sqrt{n}\|\Phi^{-1}\|\|T_1\|$. Take $C_{\hat{x}} > 1$ and $C_{\hat{u}} > 1$. Next we prove that for any given communication protocol in (2) and control protocol in (6), there exist $X(0) \in \mathscr{B}_{C_x}^{nN}$, $\hat{X}(0) \in \mathscr{B}_{C_{\hat{x}}}^{nN}$ and $\hat{U}(0) \in \mathscr{B}_{C_{\hat{u}}}^{mN}$, such that the dynamic network can not achieve cooperative stabilization, which leads to the contradiction. Denote $(\bar{\Phi} \otimes I_n)E(t)$ by $\tilde{E}_2(t)$, and the first n elements of $\tilde{E}_2(t)$ and $\tilde{\delta}_2(t)$ by $\tilde{E}_{21}(t)$ and $\tilde{\delta}_{21}(t)$, respectively, where $\bar{\Phi}$ and $\tilde{\delta}_2(t)$ are defined in the proof of Theorem 3. From (14), we have

$$\tilde{\delta}_{21}(t+1) = (A - \lambda_2(\mathcal{L})BK)\tilde{\delta}_{21}(t) + \lambda_2(\mathcal{L})BK\tilde{E}_{21}(t). \quad (20)$$

Denote $\hat{\delta}_{21}(t) = T_1^{-1}\tilde{\delta}_{21}(t)$, and let $KT_1 = (\hat{K}_3, \hat{K}_4)$ where $\hat{K}_3 \in \mathbb{R}^{m \times (n - n_{u4})}$, $\hat{K}_4 \in \mathbb{R}^{m \times n_{u4}}$. Thus, from (20), we have

$$\hat{\delta}_{21}(t+1) = \begin{pmatrix} A_{s1} - \lambda_2(\mathcal{L})B_3\hat{K}_3 & \times \\ \mathbf{0} & A_{u4} \end{pmatrix} \hat{\delta}_{21}(t)$$
$$+ \begin{pmatrix} \lambda_2(\mathcal{L})B_3\hat{K}_3 & \lambda_2(\mathcal{L})B_3\hat{K}_4 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T_1^{-1}\tilde{E}_{21}(t). \quad (21)$$

Denote the last n_{u4} elements of $\hat{\delta}_{21}(t+1)$ by $\hat{\delta}_{21_{n_{u4}}}(t+1)$. Then from (21), we have $\hat{\delta}_{21_{n_{u4}}}(t+1) = A_{u4}\hat{\delta}_{21_{n_{u4}}}(t)$. Take $X(0) = (\Phi^{-1} \otimes I_n)\left(\mathbf{0}^T, [T_1\mathbf{1}]^T\right)^T$, then $\|X(0)\|_\infty \leq \sqrt{n}\|\Phi^{-1}\|\|T_1\| < C_x$. By the definition of $\delta(t)$, and noting that π^T is the first row of Φ, we have $\delta(0) = (\Phi^{-1} \otimes I_n)\left(\mathbf{0}^T, [T_1\mathbf{1}]^T\right)^T$. Thus, $\hat{\delta}_{21}(0) = \mathbf{1}_n$ and $\hat{\delta}_{21_{n_{u4}}}(0) = \mathbf{1}_{n_{u4}}$. Take $\hat{X}(0) = \mathbf{1}_{nN}$, $\hat{U}(0) = \mathbf{1}_{mN}$, then we have $\|\hat{X}(0)\|_\infty < C_{\hat{x}}$ and $\|\hat{U}(0)\|_\infty < C_{\hat{u}}$. Since $\hat{\delta}_{21_{n_{u4}}}(0) \neq 0$, $\delta(t)$ does not vanish, which draws the contradiction. ∎

Theorem 6: For $(A, B, C, \mathscr{G})$, $L_G = +\infty$, $L_K = +\infty$ and $\varrho = 1$, suppose that for any given positive constants C_x, $C_{\hat{x}}$ and $C_{\hat{u}}$, there exist a communication protocol $H(\gamma, \alpha, \alpha_u, L, L_u, G) \in \mathscr{H}(\varrho, L_G)$ and a control protocol $U(K) \in \mathscr{U}(L_K)$, such that for any $X(0) \in \mathscr{B}_{C_x}^{nN}$, $\hat{X}(0) \in \mathscr{B}_{C_{\hat{x}}}^{nN}$ and $\hat{U}(0) \in \mathscr{B}_{C_{\hat{u}}}^{mN}$, the closed-loop system achieves inter-agent state-observation under H and U, then (A, C) is detectable.

Proof: We use the reduction to absurdity to prove this theorem. If (A, C) was not detectable, then there would exist $x_0 \in \mathbb{R}^n$, such that $CA^l x_0 = \mathbf{0}$, $l = 0, 1, 2, \ldots$, and $A^t x_0$ does not go to zero as $t \to \infty$. Take $C_x > \|x_0\|$, $C_{\hat{x}} > 0$ and $C_{\hat{u}} > 0$. Next we will prove that for any given communication protocol $H \in \mathscr{H}(1, +\infty)$ and control protocol $U \in \mathscr{U}(+\infty)$, there exist $X(0) \in \mathscr{B}_{C_x}^{nN}$, $\hat{X}(0) \in \mathscr{B}_{C_{\hat{x}}}^{nN}$ and $\hat{U}(0) \in \mathscr{B}_{C_{\hat{u}}}^{mN}$, such that the dynamic network can not achieve inter-agent state observation, which leads to the contradiction. Take $x_1(0) = x_0$ and $x_j(0) = \mathbf{0}$, $j = 2, \cdots, N$. Then $X(0) \in \mathscr{B}_{C_x}^{nN}$. By $Cx_0 = \mathbf{0}$, we have $y_j(0) = \mathbf{0}$, $j = 1, 2, \ldots, N$. Take $\hat{X}(0) = \mathbf{0}$ and $\hat{U}(0) = \mathbf{0}$, so $\hat{X}(0) \in \mathscr{B}_{C_{\hat{x}}}^{nN}$ and $\hat{U}(0) \in \mathscr{B}_{C_{\hat{u}}}^{mN}$. By (6), we know that $U(0) = \mathbf{0}$. By (3), (4), noting that $y_j(0) = \mathbf{0}$, $j = 1, 2, \ldots, N$, we know that $s_j(1) = \mathbf{0}$, $j = 1, 2, \ldots, N$, which together with $\hat{X}(0) = \mathbf{0}$ and $\hat{U}(0) = \mathbf{0}$ lead to $\hat{X}(1) = \mathbf{0}$. Then by (3), (4) and (6), it follows that $U(1) = \mathbf{0}$, and $\hat{U}(1) = \mathbf{0}$. Then by (1) and $CAx_0 = \mathbf{0}$, we have $y_j(1) = \mathbf{0}$, $j = 1, 2, \ldots, N$. Suppose that up to time t, $t = 2, 3, \ldots$, $U(k) = \hat{U}(k) = \mathbf{0}$, and $\hat{X}(k) = \mathbf{0}$, $k = 0, 1, \ldots, t-1$. Then by (1), we have $x_1(t-1) = A^{t-1}x_0$, $x_j(t-1) = \mathbf{0}$, $j = 2, 3, \ldots, N$. Noting that $CA^{t-1}x_0 = \mathbf{0}$, it follows that $y_j(t-1) = \mathbf{0}$, $j = 1, 2, \ldots, N$. And by (3), (4) and (6), we know that $\hat{X}(t) = \mathbf{0}$ and $U(t) = \hat{U}(t) = \mathbf{0}$. Then by mathematical induction, we have $\hat{X}(t) \equiv \mathbf{0}$ and $U(t) \equiv \mathbf{0}$, which together with (1) gives $x_1(t) = A^t x_0$, and $x_2(t) = \cdots = x_N(t) \equiv \mathbf{0}$. Noting that $A^t x_0$ does not go to zero as $t \to \infty$, but $\hat{X}(t) \equiv \mathbf{0}$, it follows that $E(t) = X(t) - \hat{X}(t)$ does not go to zero as $t \to \infty$, which leads to the contradiction. ∎

IV. CONCLUSION

In this note, we studied the inter-agent state observation and cooperative stabilization of discrete-time linear multi-agent systems with unmeasurable states over bandwidth limited digital networks. We proposed a class of quantized-observer based communication protocols and a class of Certainty Equivalence principle based control protocols. We showed that the simultaneous stabilizability condition and the detectability condition of agent dynamics are sufficient for the existence of communication and control protocols to ensure both the inter-agent state observation and cooperative stabilization. What's more, we proved that they are also necessary for the local and global cooperatability in some sense.

As a preliminary research, we focus on the conditions on the dynamics of agents and the network structure to ensure the existence of finite data rate inter-agent communication and control protocols. An interesting topic for future investigation is whether there is a lower bound, which is independent of the number of agents, for the communication data rate required just as the small channel capacity theorems

established in [9], [10], [12] and [16]. Note that the independency of the number of agents implies good scalability for large scale networks. The problem is more challenging. Also, Due to the time-delay, link failure or packet dropouts in networks, how to design communication and control protocols for linear multi-agent systems to ensure both the cooperative stabilization and inter-agent state observation with finite data rate, communication delay and packet dropouts is an interesting and challenging problem.

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